Goal - Category GTop.

- Classical Theorems

- Presheaves on Orb_G.

PART I GTop.

Notation : GTop = cat of G-spaces and G-maps.

- Like in classical htpy theory, we assume all spaces are CGWH, i.e. compactly generated and weak Hausdorff.

- G is assumed to be a compact Lie gp or finite gps.

Def A left G-space is a space X w/ a cts map

$$G \times X \longrightarrow X$$

$$(g, x) \longmapsto g \cdot x$$

s.t. $g_1(g_2 x) = (g_1 g_2) x$, $1_G \cdot x = x$

A right G-space is a left G-space w/ action by $g \cdot x = x g^{-1}$.

A G-map $f: X \longrightarrow Y$ is a cts map s.t. $f(g x) = g \cdot f(x)$, for all $g \in G$.

e.g. 1) $X \times Y$. G acts on it by $g(x, y) = (g \cdot x, g \cdot y)$

2) $\mathrm{Map}(X, Y) =: Y^X$. G acts on it by $(g \cdot f)(x) = g \cdot f(g^{-1} x)$.

3) $GTop(X, Y) = (Map(X, Y))^G$, G-fixed points of the mapping space.

$=: Map_G(X, Y)$.

Caution! Map_G = set of morphisms in GTop. Map = an obj in GTop.

Prop $(GTop, \times, \{*\})$ is a cartesian closed category.

pf. Suffice to find an internal hom, i.e. a functor F in GTop s.t. $F : GTop^{op} \times GTop \longrightarrow GTop$ s.t. $\forall X \in GTop$, there exist a adjoint pair $((-) \otimes X) \dashv (F(X, -)) : GTop \longrightarrow GTop$. This leads to the following proposition:

Prop $Map_G(X \times Y, Z) \cong Map_G(X, Map(Y, Z))$.

pf. Take G-fixed points of $Map(X \times Y, Z) \cong Map(X, Map(Y. Z))$

Def Pointed G-spaces $GTop_*$

consists of
- Based G-spaces: $(X, *)$, $X \in GTop$, $* \in X$, and $*$ is G-fixed.
- Based G-maps: $f : (X, *) \longrightarrow (Y, *')$, $f(*) = *'$.

Prop $Map_{*,G}(X \wedge Y, Z) \cong Map_{*,G}(X, Map_*(Y, Z))$.

FACT Pair of adjunctions:
$$\begin{tikzcd} GTop \arrow[r, shift left, "(-)_+"] \arrow[r, phantom, "\perp"] & GTop_* \arrow[l, shift left, "Forget"] \end{tikzcd}$$

Def A G-CW complex X consists of the following data:

1) G-spaces indexed by n: X^n, s.t. $X^0 = \coprod_i G/H_i$

"orbits"

where $H \subset G$ is a closed subgp.

2) Attaching maps: X^{n+1} obtained from X^n by attaching G-$(n+1)$-cells $G/H_i \times D^{n+1}$ via the map

$$G/H_i \times S^n \xrightarrow{\phi_{i,n}} X^n$$

subject to

$$\begin{tikzcd}
\coprod_{i\in I} G/H_i \times S^n \arrow[r, "\phi_n"] \arrow[d, hook] & X^n \arrow[d] \\
\coprod_{i\in I} G/H_i \times D^{n+1} \arrow[r] & X^{n+1} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Obs Regard "orbits" as points!

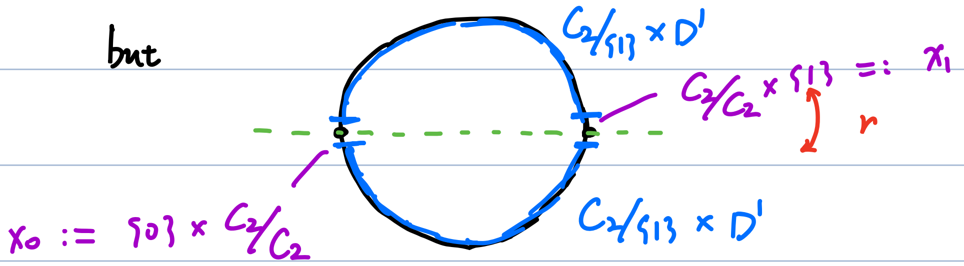
e.g. 1) $G = C_2$, $C_2 = \{1, r\}$, where r = reflection.

Now G has 2 "orbits" C_2/C_2, $C_2/\{1\}$. So

0-cells: $C_2/C_2 \times \{0\}$, $C_2/C_2 \times \{1\}$

1-cells: $C_2/\{1\} \times D^1$, $C_2/\{1\} \times D^1$

but



r fixed x_0, x_1. but takes upper $C_2/\{1\} \times D^1$ to lower $C_2/\{1\} \times D^1$. So we can identify them.

0-cells : x_0, x_1

1-cell : $C_2/\{1\} \times D^1$

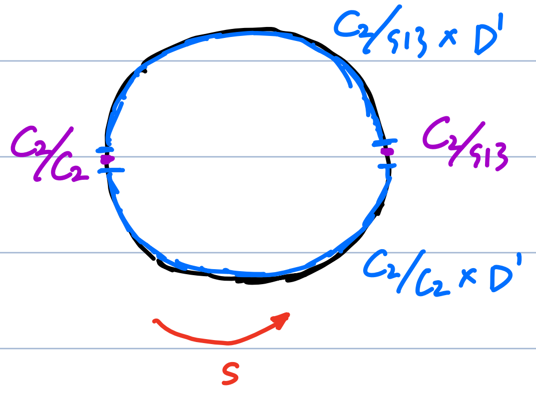
The attaching map is given by

$$\begin{tikzcd}
C_2/\{1\} \times S^0 \arrow[r, "\phi_0"] \arrow[d, hook] & X^0 = \{C_2/C_2, C_2/\{1\}\}. \arrow[d] \\
C_2/\{1\} \times D^1 \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & X^1
\end{tikzcd}$$

where $\phi_0(C_2/\{1\} \times \{0\}) = x_0$

$\phi_0(C_2/\{1\} \times \{1\}) = x_1$

2) $G = C_2 = \{1, s\}$. $s =$ rotation by 180°.



Again, s takes C_2/C_2 to $C_2/\{1\}$ and $C_2/\{1\} \times D^1$ to $C_2/C_2 \times D^1$

So

0-cell : $C_2/\{1\}$

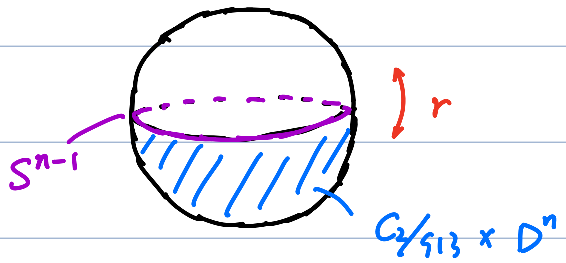
1-cell : $C_2/\{1\} \times D^1$

Attaching map :

$$\begin{tikzcd}
C_2/\{1\} \times S^0 \arrow[r, "\phi_0"] \arrow[d, hook] & X^0 = \{C_2/\{1\}\}. \arrow[d] \\
C_2/\{1\} \times D^1 \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & X^1
\end{tikzcd}$$

where $\phi_0(C_2/\{1\} \times \{0\}) = C_2/\{1\}$. and we identify $C_2/\{1\} \sim C_2/C_2$.

3) $G = C_2 = \{1, r\}$, $r =$ reflection. $X = S^n$.



Similar to (1).

n-cell : $C_2/\{1\} \times D^n$

Note two d-cells $C_2/C_2 \times D^d$ for each $0 \le d < n$.

Attaching map :

$$\begin{tikzcd}
C_2/\{1\} \times S^{n-1} \arrow[r, "\phi_{n-1}"] \arrow[d, hook] & X^{n-1} \arrow[d] \\
C_2/\{1\} \times D^n \arrow[r] & X^n \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

where $\phi_{n-1}(C_2/\{1\} \times S^{n-1}) = S^{n-1}$.

Def A G-homotopy between $f, g : X \to Y$ in GTop is a G-map $H : X \times I \to Y$ s.t. $f = H(-, 0)$, $g = H(-, 1)$ w/ trivial G-action. f is a G-homotopy equivalence if $\exists f^{-1} : Y \to X$ s.t. $f \circ g$, $g \circ f$ are homotopic to identity at corresponding targets.

Notation $[X, Y]_G =:$ htpy class of G-maps.

Def Let $H \subset G$ be closed subgp. The H-equivariant htpy gps are

$$\pi_n^H(X) = \pi_0 \operatorname{Hom}_G(G/H_+ \wedge S^n, X) = \pi_n(X^H)$$

Def A weak (htpy) equivalence is a G-map $f : X \to Y$ s.t. $f^H : X^H \to Y^H$ is weak equivalence, $\forall H \subset G$ closed.

Rk $X^H = \{ x \in X : hx = x \text{ for all } h \in H \}$. (categorical) fixed pts.

$X_H = X/H = X/(x \sim hx)$ orbit space

both are $W_G H$ - spaces . where $W_G H$ is the Weyl gp .

$W_G H = N_G H / H$

Prop Pair of adjoint functors :

$(-)^{\text{triv}}$:

$$\begin{tikzcd} Top \arrow[r, "\perp", "\perp"'] & GTop \arrow[l, bend right, "(-)_G"'] \arrow[l, bend left, "(-)^G"] \end{tikzcd}$$

w/ trivial G-action

PART II Classical Theorems.

Now we assume all $X \in GTop$ are G-CW complexes.

Def Let $\nu : C(G) \longrightarrow \mathbb{N}$.

conjugacy classes of G.

Call $f: X \to Y$ is a ν-equivalence if $f^H : X^H \longrightarrow Y^H$ is a $\nu(H)$-equivalence for all H (i.e. $\pi_n^H(f)$ is bijection when $n < \nu(H) - 1$; and surjection when $n = \nu(H)$.)

- Particularly . if $\nu = \text{const}$. then this is the classical n-equivalence.

Thm (Homotopy extension & lifting property, aka. HELP)

Let $A \overset{i}{\hookrightarrow} X$ be a subcomplex. $\dim X \leq \nu$. Let $f: Y \to Z$ be an ν-equivalence. Given $g: A \to Y$, $H: A \times I \to Z$, $h: X \to Z$ w/ the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, hook, "j_1"] \arrow[d, "i"'] & A \times I \arrow[d, "H"'] & A \arrow[l, hook', "j_2"'] \arrow[d, "g"] \\
X \arrow[r, "h"] & Z & Y \arrow[l, "f"']
\end{tikzcd}$$

Then $\exists\ \tilde{g}$, $\widetilde{H}$ lifts g, H, respectively:

$$\begin{tikzcd}
A \arrow[rr, "j_1"] \arrow[dd, "i"'] & & A \times I \arrow[dl, "H"'] \arrow[dd, "i \times id"] \\
& Z & \\
X \arrow[ur, "f"] \arrow[rr, "j_1'"'] & & X \times I \arrow[ul, dashed, red, "\widetilde{H}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rrr, "j_2"] \arrow[dr, "g"] \arrow[dd, "i"'] & & & A \times I \arrow[dl, "H"'] \arrow[dd, "i \times id"] \\
& Y \arrow[r, "f"] & Z & \\
X \arrow[ur, dashed, red, "\tilde{g}"] \arrow[rrr, "j_2'"'] & & & X \times I \arrow[ul, dashed, red, "\widetilde{H}"']
\end{tikzcd}$$

Prop Let $f: Y \to Z$ be an ν-equivalence, and it induces ($X \in GTop$)

$$f_*: [X, Y]_G \to [X, Z]_G$$

Then f_* is a $\begin{cases} \text{bijection}, & \text{if } \dim X < \nu \\ \text{surjection}, & \text{if } \dim X = \nu \end{cases}$

Cor (Whitehead)

Let $f: Y \to Z$ be an ν-equivalence, $\dim Y, Z < \nu$. then f is a G-homotopy equivalence.

Thm (CW approximation)

Let $X \in GTop$. Then $\exists$ G-CW complex $\tilde{X}$ and a weak equivalence $\tilde{X} \longrightarrow X$. This $\widetilde{(-)}$ is functorial.

PART III Presheaves on Orb_G.

Model Structure on $GTop$ (or $GTop_*$) :

1. Weak equivalences :

   $f : X \to Y$ s.t. $f^H : X^H \longrightarrow Y^H$ is weak equivalence, $\forall$ $H \subset G$ closed.

2. Fibrations :

   $f : X \to Y$ s.t. $f^H : X^H \longrightarrow Y^H$ is a fibration, $\forall$ $H \subset G$ closed.

3. Cofibrations :

   $f : X \longrightarrow Y$ s.t. it has left lifting property w.r.t. acyclic fibrations.

Def The orbit category Orb_G :

- obj : G/H. $H \subset G$ closed subgp
- mor : G-equivariant maps.

Notation $\mathcal{P}(Orb_G)$ = cat of presheaves on Orb_G.

FACT $\mathcal{P}(Orb_G)$ has model structure :

1. Weak equivalences :

$\eta : F_1 \Rightarrow F_2$ s.t. $\eta_{G/H} : F_1(G/H) \longrightarrow F_2(G/H)$ is a w.e.

2. Fibrations :

$\eta : F_1 \Rightarrow F_2$ s.t. $\eta_{G/H} : F_1(G/H) \longrightarrow F_2(G/H)$ is a fibration.

3. Cofibrations :

$\eta : F_1 \Rightarrow F_2$ s.t. it has left lifting property w.r.t. acyclic fibrations.

Thm. (Elmendorf)

There's a Quillen equivalence

$$\Psi : \mathcal{P}(Orb_G) \underset{\perp}{\rightleftarrows} GTop : \Phi$$

where $\Psi(F) = F(G/e)$, the action is determined by $Aut(G)$

$\Phi(X) = L$, w/ $L(G/H) = X^H$.

Here X^H has a natural Weyl gp action.

Application Eilenberg - MacLane G - spaces.

Bredon (co)homology theory.

Rk. Both model structures on $\mathcal{P}(Orb_G)$ & $GTop$ are cofibrantly generated.

- More comments on "base-change functor":

Let $f: H \longrightarrow K$ be a gp homomorphism for $H, K \subset G$ as closed subgps. Then there are pairs of adjunctions:

$$\begin{tikzcd}
K\text{Top} \arrow[r, "f^*"] & H\text{Top} \arrow[l, bend right=50, "f_!"'] \arrow[l, bend left=50, "f_*"]
\end{tikzcd}$$

$\perp$ $\perp$

where $f_!(X) = K \times_H X$

$f_*(X) = (Map(K, X))^H$

f^* is induced by f.

e.g. 1) $K = \{e\}$, $f: G \longrightarrow \{e\}$. $f^* = (-)^{triv}$

$\Rightarrow f_!(X) = * \times_G X = X/G$

$f_*(X) = Map(*, X) = X^G$.

2) $f: H \hookrightarrow K$, then $f^* = Res_H^K$ restriction,

$\Rightarrow f_!(X) = K \times_H X$

$f_*(X) = Map_H(K, X)$

- Modern viewpoint

Let BG = small category. G is the gp of interest.

obj $= *$

mor $= G$, G acts on $*$

Then $G\text{Top} \cong Fun(BG, Top)$, via the identification

functor category

$X \longleftrightarrow (F_X : BG \longrightarrow Top)$. $\quad F_X(\underbrace{* \curvearrowleft G}_{G\text{-action}}) = X \curvearrowleft G$.

Regard BG as a small diagram, then

$$X^G = \lim_{BG} F_X . \qquad X_G = \operatorname{colim}_{BG} F_X .$$

▶ Rewrite the previous generalization as follows :

For $f : H \longrightarrow K$, $\quad H, K \subset G$, closed.

It corresponds to $\tilde{f} : BH \longrightarrow BK$. Then

$$\begin{tikzcd}
Fun(BK, Top) \arrow[r, "\tilde{f}^*"] \arrow[d, phantom, "\cong" rotate=90] & Fun(BH, Top) \arrow[d, phantom, "\cong" rotate=90] \\
KTop \arrow[r, "f^*"] & HTop
\end{tikzcd}$$

$\tilde{f}^*$ has left (resp. right) adjoint, given by the left (resp. right) Kan extension. Namely,

$$\begin{tikzcd}
BH \arrow[r, "F"] \arrow[d, "\tilde{f}"'] & Top \\
BK \arrow[ur, dashed] &
\end{tikzcd}$$

$$\Rightarrow \quad Lan_{\tilde{f}} F = f_! F$$
$$Ran_{\tilde{f}} F = f_* F$$

When $K = \{e\}$, $BK = * \overset{id}{\curvearrowleft}$. $\quad F : BH \longrightarrow *$

$$\Rightarrow \quad Lan_{\tilde{f}} F = \operatorname{colim}_{BH} F . \qquad Ran_{\tilde{f}} F = \lim_{BH} F .$$